Universität Hamburg
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CHAI
Humanities-Centered AI

Transformer-based Algorithmics

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June 10, 2026

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Topics for Today

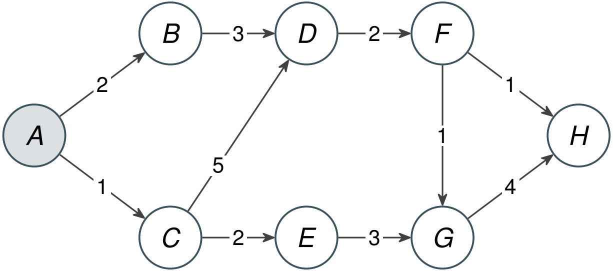
- Shortest paths in directed acyclic graphs (DAGs)

Acknowledgement

- Some of the upcoming content is taken from the following lectures and corresponding exercises and has been translated and partially modified:
  - Prof. Dr. Ralf Möller: »Algorithmen und Datenstrukturen – Vorlesung«
  - Dr. Magnus Bender: »Algorithmen und Datenstrukturen – Übung«

Exercise 1 – Shortest Paths in DAGs

Consider the weighted, directed acyclic graph (DAG) $G = (V, E)$ below with source A.



(a) Compute a **topological order** of the vertices using a FIFO queue (state the queue and the assigned numbers after each step).

(b) Use the topological order to compute the **shortest distances** $d(v)$ from A to every other vertex.

(c) Give the **shortest path $A \to H$**.

Convention: when processing a vertex, mark its outgoing edges in alphabetical order of the target.

Shortest Paths in DAGs – Reminder

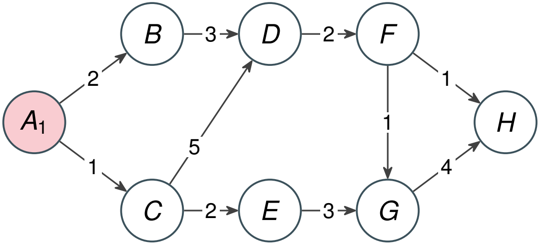
DAG Shortest Paths in a Nutshell

On a DAG with arbitrary edge weights, single-source shortest paths from s can be computed in $O(|V| + |E|)$:

(1) Compute a **topological order** of the vertices.
(2) Initialize $d(s) = 0$ and $d(v) = \infty$ for $v \neq s$.
(3) Process the vertices in topological order; for each u, **relax** every outgoing edge $u \to v$: $d(v) = \min\{d(v),\ d(u) + w(u, v)\}$.

Topological sort (FIFO version). Initialize a FIFO queue q with every *source* (in-degree 0), and a counter $n = 1$. When inserting v into q, set $\mathrm{num}(v) = n$ and increment n. While q is non-empty: pop u, mark every outgoing edge $u \to v$; whenever *all* in-edges of v are marked, insert v into q.

Exercise 1 – (a) Topological Sort Trace

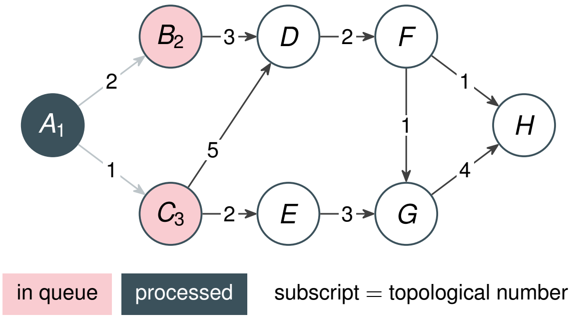


Init. Source A has in-degree 0; insert it with $\mathrm{num}(A){=}1$.

$q = [A]$

in queue | processed | subscript = topological number

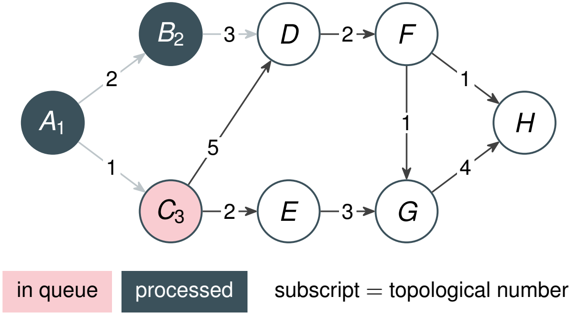
Exercise 1 – (a) Topological Sort Trace



Step 1. Pop A; mark $A \to B$, then $A \to C$. Both B and C now have all in-edges marked: insert with $\mathrm{num}(B) = 2$, $\mathrm{num}(C) = 3$.

$q = [\, B,\ C \,]$

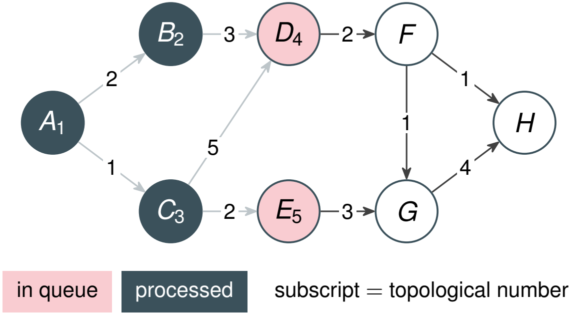
Exercise 1 – (a) Topological Sort Trace



Step 2. Pop B; mark $B \to D$. D still has an unmarked in-edge from C.

$q = [\,C\,]$

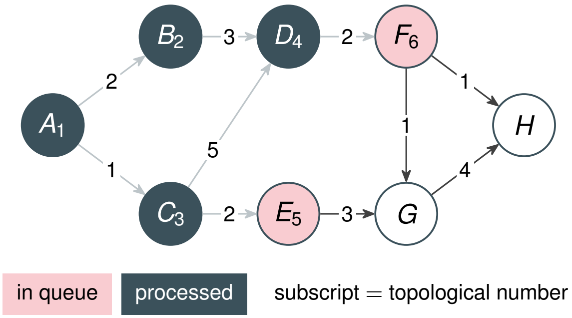
Exercise 1 – (a) Topological Sort Trace



Step 3. Pop C; mark $C \to D$ (so D is complete) and $C \to E$ (so E is complete). Insert D with $\mathrm{num}(D) = 4$, then E with $\mathrm{num}(E) = 5$.

$q = [\, D,\ E \,]$

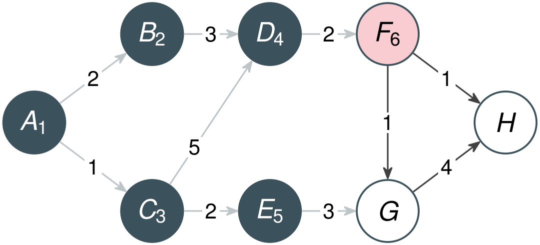
Exercise 1 – (a) Topological Sort Trace



Step 4. Pop D; mark $D \to F$. F is complete: insert with $\mathrm{num}(F) = 6$.

$q = [\,E,\ F\,]$

Exercise 1 – (a) Topological Sort Trace

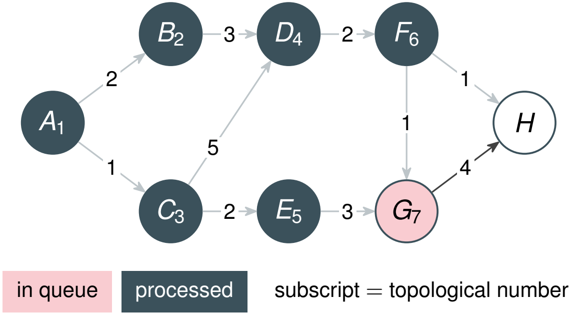


in queue | processed | subscript = topological number

Step 5. Pop E; mark $E \to G$. G still has an unmarked in-edge from F.

$q = [\,F\,]$

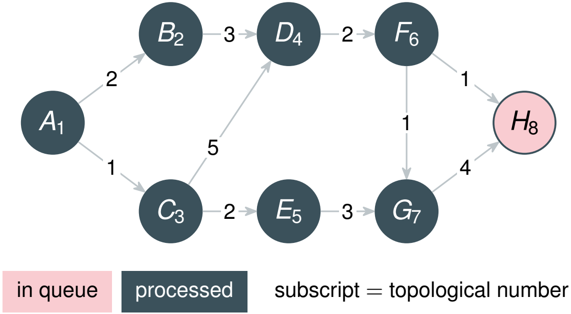
Exercise 1 – (a) Topological Sort Trace



Step 6. Pop F; mark $F \to G$ (so G is complete) and $F \to H$. Insert G with $\mathrm{num}(G) = 7$.

$q = [\, G \,]$

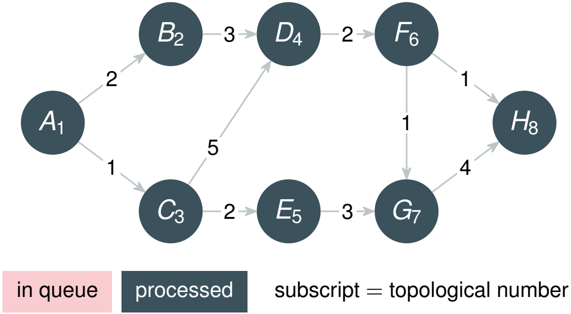
Exercise 1 – (a) Topological Sort Trace



Step 7. Pop G; mark $G \to H$ (so H is complete). Insert H with $\mathrm{num}(H) = 8$.

$q = [\,H\,]$

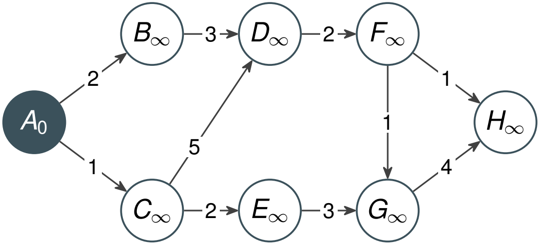
Exercise 1 – (a) Topological Sort Trace



Step 8. Pop H; no outgoing edges. $q = [\,]$ – **done**.

Topological order:
A, B, C, D, E, F, G, H

Exercise 1 – (b) Distance Updates in Topological Order

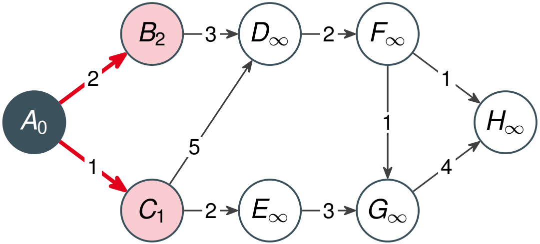


Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Process u	Relaxations
	(init)

Init: $d(A)=0$, $d(v)=\infty$ for $v \neq A$.

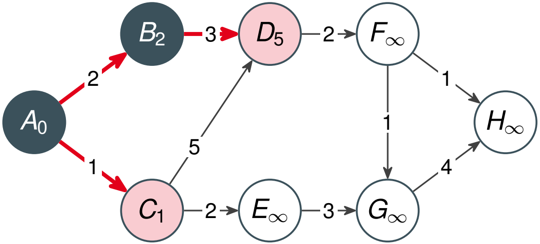
Exercise 1 – (b) Distance Updates in Topological Order



Process u	Relaxations
A $(d=0)$	$d(B)=2$, $d(C)=1$

Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

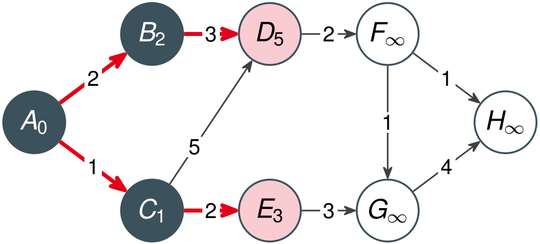
Exercise 1 – (b) Distance Updates in Topological Order



Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Process u	Relaxations
A $(d{=}0)$	$d(B){=}2$, $d(C){=}1$
B $(d{=}2)$	$d(D){=}2{+}3{=}5$

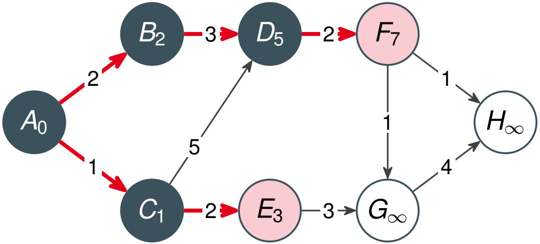
Exercise 1 – (b) Distance Updates in Topological Order



Process u	Relaxations
A $(d{=}0)$	$d(B){=}2,\quad d(C){=}1$
B $(d{=}2)$	$d(D){=}2{+}3{=}5$
C $(d{=}1)$	$C{\to}D$: $1{+}5{=}6 \not< 5$
	$d(E){=}1{+}2{=}3$

Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

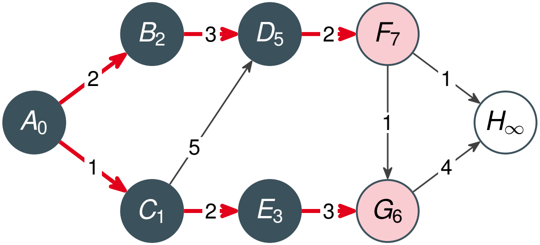
Exercise 1 – (b) Distance Updates in Topological Order



Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Process u	Relaxations
A $(d{=}0)$	$d(B){=}2$, $d(C){=}1$
B $(d{=}2)$	$d(D){=}2{+}3{=}5$
C $(d{=}1)$	$C{\to}D$: $1{+}5{=}6 \not< 5$
	$d(E){=}1{+}2{=}3$
D $(d{=}5)$	$d(F){=}5{+}2{=}7$

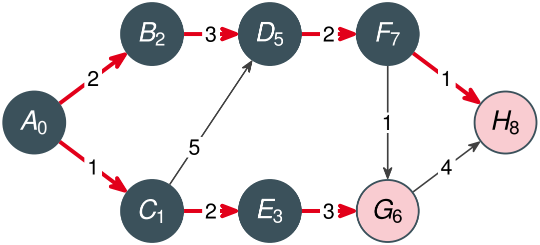
Exercise 1 – (b) Distance Updates in Topological Order



Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Process u	Relaxations
A $(d{=}0)$	$d(B){=}2, \quad d(C){=}1$
B $(d{=}2)$	$d(D){=}2{+}3{=}5$
C $(d{=}1)$	$C{\to}D$: $1{+}5{=}6 \not< 5$
	$d(E){=}1{+}2{=}3$
D $(d{=}5)$	$d(F){=}5{+}2{=}7$
E $(d{=}3)$	$d(G){=}3{+}3{=}6$

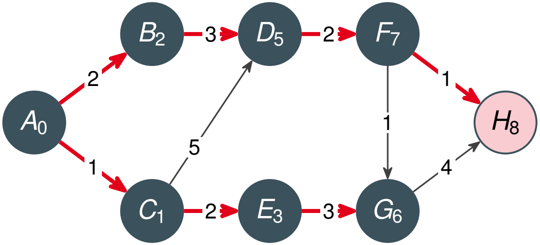
Exercise 1 – (b) Distance Updates in Topological Order



Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Process u	Relaxations
$A\ (d=0)$	$d(B)=2,\quad d(C)=1$
$B\ (d=2)$	$d(D)=2+3=5$
$C\ (d=1)$	$C \to D$: $1+5=6 \not< 5$
	$d(E)=1+2=3$
$D\ (d=5)$	$d(F)=5+2=7$
$E\ (d=3)$	$d(G)=3+3=6$
$F\ (d=7)$	$F \to G$: $7+1=8 \not< 6$
	$d(H)=7+1=8$

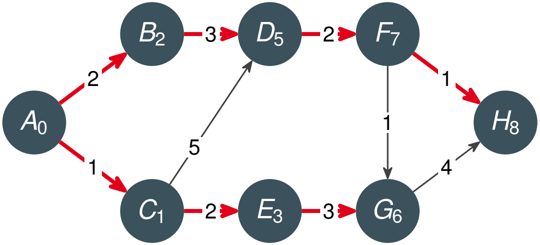
Exercise 1 – (b) Distance Updates in Topological Order



Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Process u	Relaxations
A $(d{=}0)$	$d(B){=}2,\quad d(C){=}1$
B $(d{=}2)$	$d(D){=}2{+}3{=}5$
C $(d{=}1)$	$C{\to}D$: $1{+}5{=}6 \not< 5$
	$d(E){=}1{+}2{=}3$
D $(d{=}5)$	$d(F){=}5{+}2{=}7$
E $(d{=}3)$	$d(G){=}3{+}3{=}6$
F $(d{=}7)$	$F{\to}G$: $7{+}1{=}8 \not< 6$
	$d(H){=}7{+}1{=}8$
G $(d{=}6)$	$G{\to}H$: $6{+}4{=}10 \not< 8$

Exercise 1 – (b) Distance Updates in Topological Order



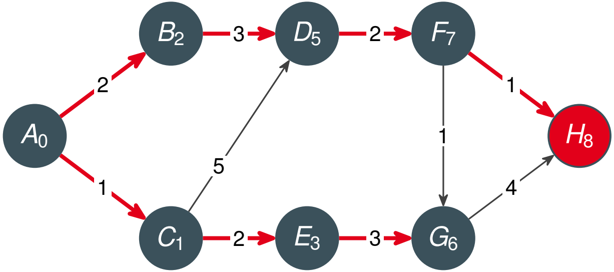
Subscripts show the current $d(v)$. **Red** edges form the shortest-path tree.

Final distances:

$d(A)=0,\ d(B)=2,\ d(C)=1,\ d(D)=5,$
$d(E)=3,\ d(F)=7,\ d(G)=6,\ d(H)=8.$

Process u	Relaxations
$A\ (d=0)$	$d(B)=2,\quad d(C)=1$
$B\ (d=2)$	$d(D)=2+3=5$
$C\ (d=1)$	$C\to D$: $1+5=6 \not< 5$
	$d(E)=1+2=3$
$D\ (d=5)$	$d(F)=5+2=7$
$E\ (d=3)$	$d(G)=3+3=6$
$F\ (d=7)$	$F\to G$: $7+1=8 \not< 6$
	$d(H)=7+1=8$
$G\ (d=6)$	$G\to H$: $6+4=10 \not< 8$
$H\ (d=8)$	(no outgoing edges)

Exercise 1 – (c) Result: Shortest-Path Tree from A



Vertex labels v_d show the final distance from A; **red** edges form the **shortest-path tree**.

Shortest path $A \to H$:

$$A \xrightarrow{2} B \xrightarrow{3} D \xrightarrow{2} F \xrightarrow{1} H$$

Total length: $2 + 3 + 2 + 1 = 8$