# Lifted Model Construction under Approximate Commutativity

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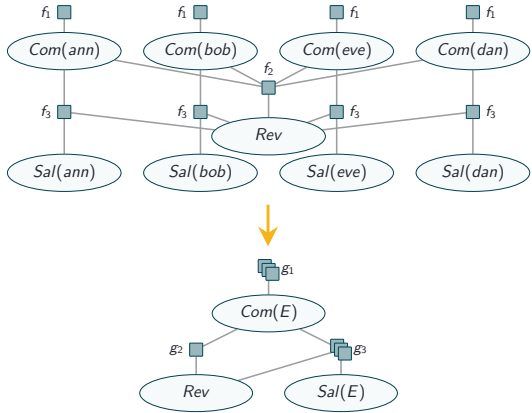
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## What is this Work About?

- Probabilistic inference in a factor graph is *intractable* in general.
- *Lifted inference* exploits indistinguishability of objects for tractable inference.
- Lifted model construction involves the compression of *commutative factors*.

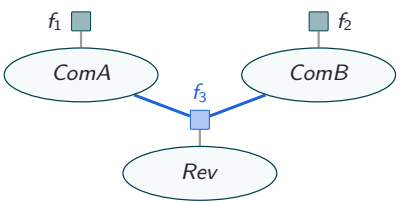


# Commutative Factors: Definition

**Commutative Factor (Luttermann et al., 2024)**

A factor $\phi(R_1, \dots, R_n)$ is **commutative** w.r.t. $\boldsymbol{C} \subseteq \{R_1, \dots, R_n\}$ if its output is invariant under *every* permutation of the arguments in $\boldsymbol{C}$:

$$\phi(r_1, \dots, r_n) = \phi(r_{\pi(1)}, \dots, r_{\pi(n)}) \quad \text{for all } \pi \text{ fixing } R_i \notin \boldsymbol{C}.$$



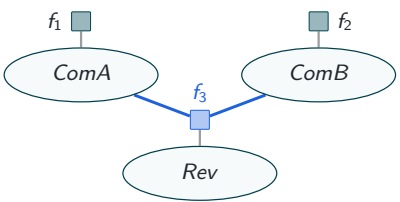

| *ComA* | *ComB* | *Rev* | $\phi_3(\mathit{ComA}, \mathit{ComB}, \mathit{Rev})$ |
|---|---|---|---|
| *high* | *high* | *high* | $\varphi_1$ |
| *high* | *high* | *low* | $\varphi_2$ |
| *high* | *low* | *high* | $\varphi_3$ |
| *high* | *low* | *low* | $\varphi_4$ |
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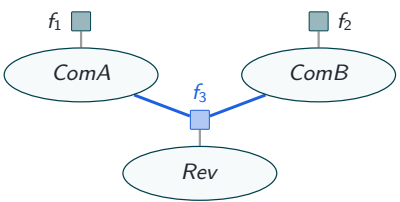

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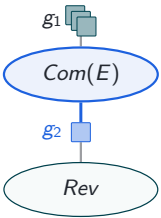

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# Commutative Factors: Compression

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| $[2, 0]$ | *high* | $\varphi_1$ |
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## Problem: Small Deviations Break Commutativity

- Learning from data results in *inaccurate estimates*.
- Values that *should* coincide can slightly differ: $\varphi_3 \neq \varphi_3'$.
- Strict equality fails $\Rightarrow$ no commutativity, *no lifting*.

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| *ComA* | *ComB* | *Rev* | $\phi_3(ComA, ComB, Rev)$ |
|---|---|---|---|
| *high* | *high* | *high* | 0.91 |
| *high* | *high* | *low* | 0.17 |
| *high* | *low* | *high* | 0.72 |
| *high* | *low* | *low* | 0.34 |
| *low* | *high* | *high* | 0.71 |
| *low* | *high* | *low* | 0.34 |
| *low* | *low* | *high* | 0.14 |
| *low* | *low* | *low* | 0.83 |

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**Observation**

The symmetry is clearly present from a modelling perspective: the definition of commutativity is simply too strict.

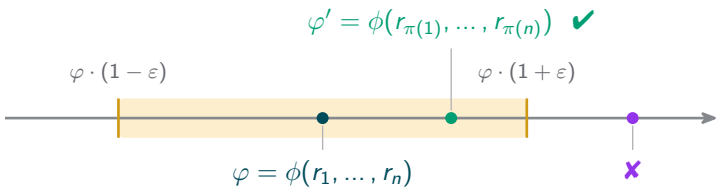
$\Rightarrow$ *Soften the underlying equality requirement.*

# Approximate Equality: $\varepsilon$-Equivalence

**$\varepsilon$-Equivalence (Luttermann et al., 2025)**

Potentials $\varphi_1, \varphi_2 \in \mathbb{R}_{\geq 0}$ are **$\varepsilon$-equivalent**, written $\varphi_1 =_\varepsilon \varphi_2$, if

$$\varphi_1 \in [\varphi_2 \cdot (1-\varepsilon), \varphi_2 \cdot (1+\varepsilon)] \quad \text{and} \quad \varphi_2 \in [\varphi_1 \cdot (1-\varepsilon), \varphi_1 \cdot (1+\varepsilon)].$$

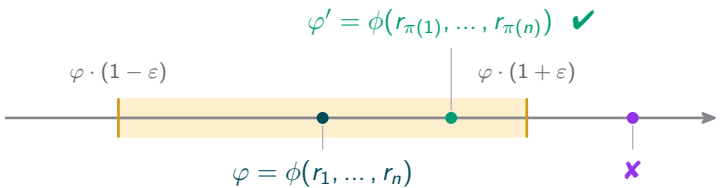


# Approximate Commutativity: $\varepsilon$-Commutativity

## $\varepsilon$-Commutative Factor

A factor $\phi(R_1, \dots, R_n)$ is **$\varepsilon$-commutative** w.r.t. $\boldsymbol{C} \subseteq \{R_1, \dots, R_n\}$ if its output is approximately invariant under *every* permutation of the arguments in $\boldsymbol{C}$:

$$\phi(r_1, \dots, r_n) =_\varepsilon \phi(r_{\pi(1)}, \dots, r_{\pi(n)}) \quad \text{for all } \pi \text{ fixing } R_i \notin \boldsymbol{C}.$$

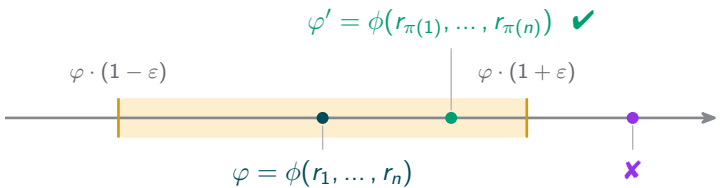


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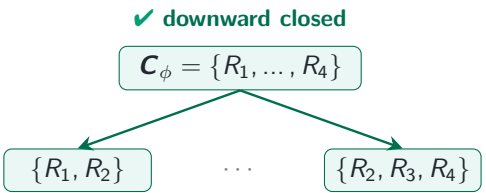
$$\phi(r_1, \dots, r_n) =_\varepsilon \phi(r_{\pi(1)}, \dots, r_{\pi(n)}) \quad \text{for all } \pi \text{ fixing } R_i \notin \boldsymbol{C}.$$

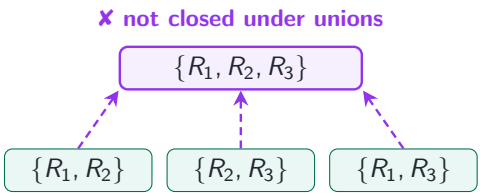



- For $\varepsilon = 0$, $\varepsilon$-commutativity reduces to strict commutativity.

## Structural Properties of $\varepsilon$-Commutativity

- No transitivity: $\{R_1, R_2\}$ and $\{R_2, R_3\}$ does *not* imply $\{R_1, R_3\}$.
- Efficient algorithms to compute maximum sized commutative subsets (Luttermann et al., 2026) are no longer applicable.



## Compression via Symmetrised Representatives

| *ComA* | *ComB* | $\phi(\cdot,\cdot,\mathit{high})$ |
|---|---|---|
| *high* | *high* | $\varphi_1$ |
| *high* | *low* | $\varphi_3$ |
| *low* | *high* | $\varphi_3'$ |
| *low* | *low* | $\varphi_5$ |

symmetry set $\mathfrak{S}_\phi(\boldsymbol{r})$

**symmetrise**

$\varphi_3^* = \frac{1}{2}(\varphi_3 + \varphi_3')$

| $\#_E[Com(E)]$ | $\phi^*(\cdot,\mathit{high})$ |
|---|---|
| $[2,0]$ | $\varphi_1^*$ |
| $[1,1]$ | $\varphi_3^*$ |
| $[0,2]$ | $\varphi_5^*$ |

one representative

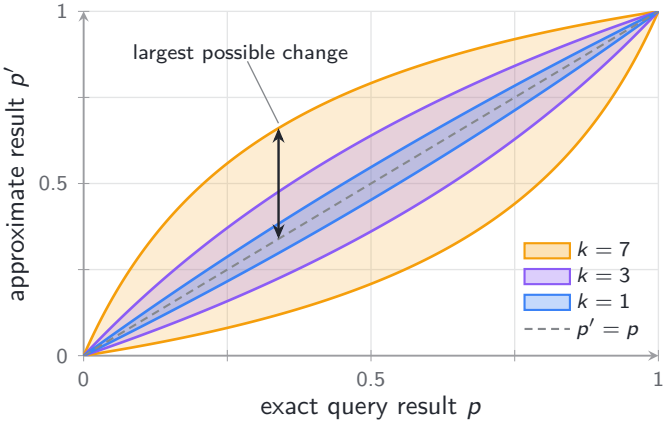
- Symmetrised representative factor:

$$\phi^*(\boldsymbol{r}) := \sum_{\boldsymbol{r}' \in \mathfrak{S}_\phi(\boldsymbol{r})} \omega_{\boldsymbol{r}'} \cdot \phi(\boldsymbol{r}') \quad \text{with} \quad \omega_{\boldsymbol{r}'} \geq 0 \quad \text{and} \quad \sum_{\boldsymbol{r}'} \omega_{\boldsymbol{r}'} = 1.$$

- Guarantee: $\phi$ and $\phi^*$ are $\varepsilon$-equivalent for *every* admissible weighting.
- The arithmetic mean minimises the squared deviation between $\phi$ and $\phi^*$.

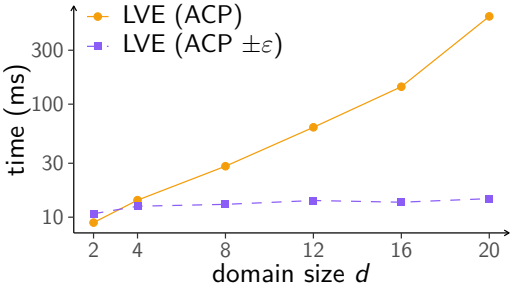
## Bounds on the Approximation Error

- Replacing $k$ $\varepsilon$-commutative factors by representatives keeps *every* query result within a guaranteed range, e.g., for $\varepsilon = 0.1$:

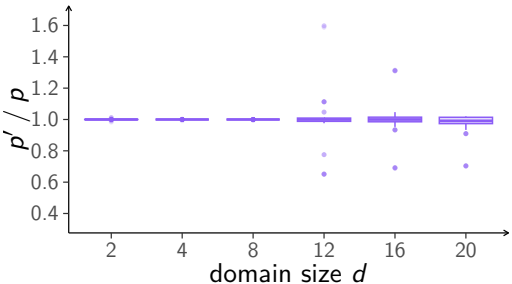


## Experiments

- Substantially faster inference due to the compression of $\varepsilon$-commutative factors.
- Query results $p'$ remain close to the true query results $p$.



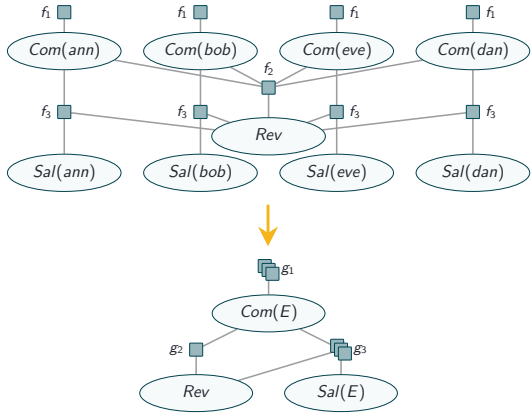

Avg. run time of lifted variable elimination.




Per-query quotient $p' / p$ distribution.

## Conclusion

- Introduction of $\varepsilon$-commutativity to tolerate small numerical deviations naturally occurring in practice.
- Compression by symmetrised representatives with sharp error bounds for arbitrary queries.
- Experiments show that the actual error remains well below the theoretical bounds in practice.



## References I


- Luttermann, Malte, Tanya Braun, Ralf Möller, and Marcel Gehrke (2024). "Colour Passing Revisited: Lifted Model Construction with Commutative Factors". In: *Proceedings of the Thirty-Eighth AAAI Conference on Artificial Intelligence (AAAI-2024)*. AAAI Press, pp. 20500–20507.
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- Luttermann, Malte, Jan Speller, Marcel Gehrke, Tanya Braun, Ralf Möller, and Mattis Hartwig (2025). "Approximate Lifted Model Construction". In: *Proceedings of the Thirty-Fourth International Joint Conference on Artificial Intelligence (IJCAI-2025)*. IJCAI Organization, pp. 9077–9085.