Approximate Lifted Model Construction

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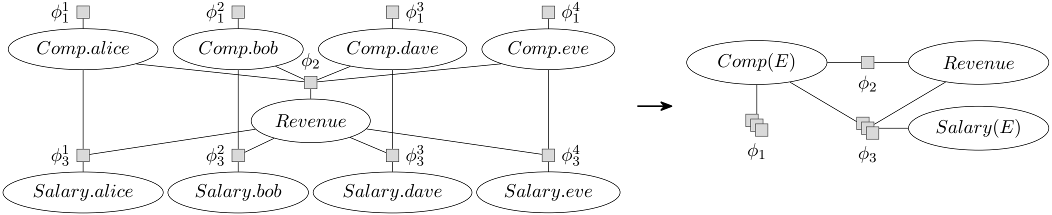
Problem Setup

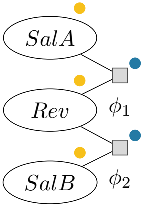
- Input: A factor graph G
- Output: A parametric factor graph with approximately equivalent semantics as G
  - With a minimal approximation error
  - With theoretical guarantees for the change in query results



Previous Work: The Advanced Colour Passing (ACP) Algorithm

(Kersting, Ahmadi, and Natarajan, 2009; Ahmadi et al., 2013; Luttermann et al., 2024)

- Assign colours to random variables according to their ranges and evidence
- Assign colours to factors according to their potential tables
- Pass colours around to detect symmetries in the graph



$SalA$	Rev	$\phi_1(SalA, Rev)$
high	high	φ_1
high	low	φ_2
low	high	φ_3
low	low	φ_4

$SalB$	Rev	$\phi_2(SalB, Rev)$
high	high	φ_1
high	low	φ_2
low	high	φ_3
low	low	φ_4

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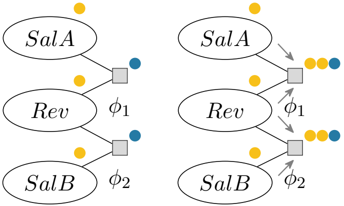

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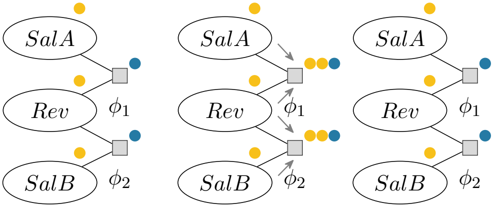

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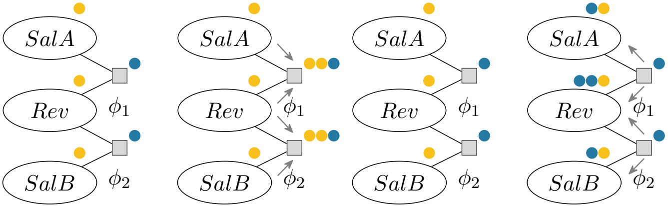

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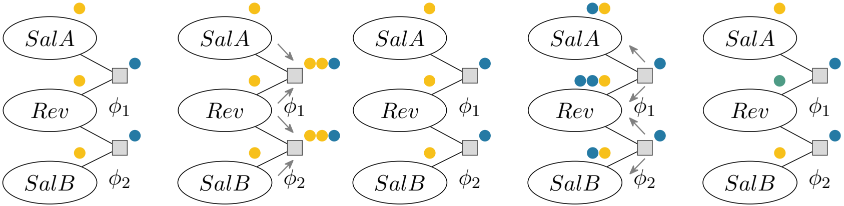

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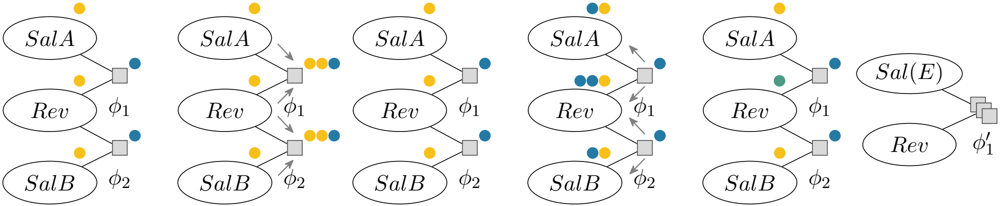

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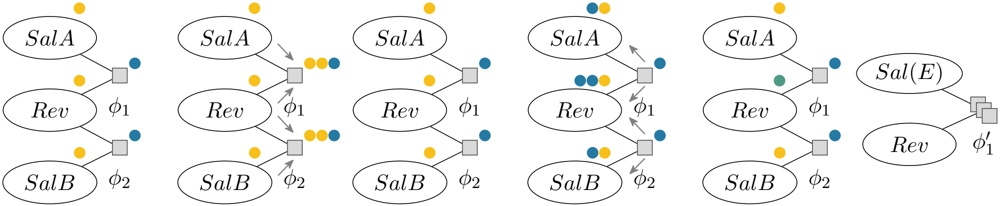
high	high	φ_1
high	low	φ_2
low	high	φ_3
low	low	φ_4

$Sal(E)$	Rev	$\phi_1'(Sal(E), Rev)$
high	high	φ_1
high	low	φ_2
low	high	φ_3
low	low	φ_4

Limitation of the Advanced Colour Passing Algorithm

- Potentials of factors must be strictly equal for factors to receive the same colour
- Limits the applicability of ACP in practice
  - E.g., if potentials are estimated from data



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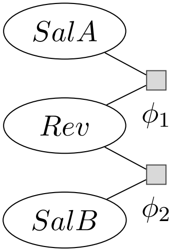
high	high	φ_1
high	low	φ_2
low	high	φ_3
low	low	φ_4

$Sal(E)$	Rev	$\phi'_1(Sal(E), Rev)$
high	high	φ_1
high	low	φ_2
low	high	φ_3
low	low	φ_4

Approximate Lifted Model Construction

- So far: Strict equality between potentials required
  - E.g., $\varphi_1 = \varphi_1$, $\varphi_2 = \varphi_2$, $\varphi_3 = \varphi_3$, $\varphi_4 = \varphi_4$



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high	high	φ_1
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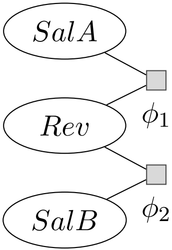
low	high	φ_3
low	low	φ_4

Approximate Lifted Model Construction

- Our goal: Allow for a small deviation between potentials for practical applicability
  - E.g., $\varphi_1 \approx \varphi'_1$, $\varphi_2 \approx \varphi'_2$, $\varphi_3 \approx \varphi'_3$, $\varphi_4 \approx \varphi'_4$



$SalA$	Rev	$\phi_1(SalA, Rev)$
high	high	φ_1
high	low	φ_2
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$SalB$	Rev	$\phi_2(SalB, Rev)$
high	high	φ'_1
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low	high	φ'_3
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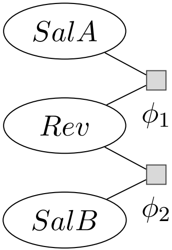

$SalA$	Rev	$\phi_1(SalA, Rev)$
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$SalB$	Rev	$\phi_2(SalB, Rev)$
high	high	φ'_1
high	low	φ'_2
low	high	φ'_3
low	low	φ'_4

- How much deviation should be allowed and what is the impact on query results?

ε-Equivalence

- Potentials $\varphi_1 \in \mathbb{R}^+$ and $\varphi_2 \in \mathbb{R}^+$ are ε-*equivalent* if

$$\varphi_1 \in [\varphi_2 \cdot (1-\varepsilon), \varphi_2 \cdot (1+\varepsilon)] \text{ and}$$
$$\varphi_2 \in [\varphi_1 \cdot (1-\varepsilon), \varphi_1 \cdot (1+\varepsilon)]$$

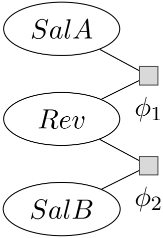
- Factors $\phi_1(R_1, \ldots, R_n)$ and $\phi_2(R'_1, \ldots, R'_n)$ are ε-equivalent if all potentials in their potential tables are ε-equivalent
- E.g., for $\varepsilon = 0.1$, $\phi_1(SalA, Rev)$ and $\phi_2(SalB, Rev)$ are ε-equivalent:

$SalA$	Rev	$\phi_1(SalA, Rev)$
high	high	0.81
high	low	0.32
low	high	0.51
low	low	0.21

$SalB$	Rev	$\phi_2(SalB, Rev)$
high	high	0.84
high	low	0.31
low	high	0.51
low	low	0.20

Grouping ε-Equivalent Factors

- A representative potential table is required to construct a lifted representation



$SalA$	Rev	$\phi_1(SalA, Rev)$
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low	high	φ'_3
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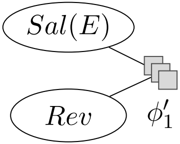
↓



$Sal(E)$	Rev	$\phi'_1(Sal(E), Rev)$
high	high	φ^*_1
high	low	φ^*_2
low	high	φ^*_3
low	low	φ^*_4

Grouping ε-Equivalent Factors

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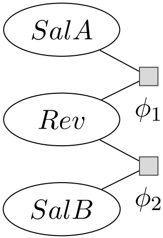

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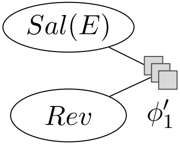

$Sal(E)$	Rev	$\phi'_1(Sal(E), Rev)$
high	high	φ^*_1
high	low	φ^*_2
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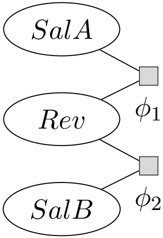
low	low	φ^*_4

- How to choose φ^*_i?

Minimisation of the Approximation Error

- Choose φ_i^* as the row-wise arithmetic mean of the potentials



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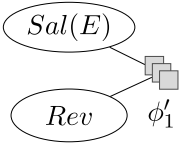
low	high	φ_3'
low	low	φ_4'

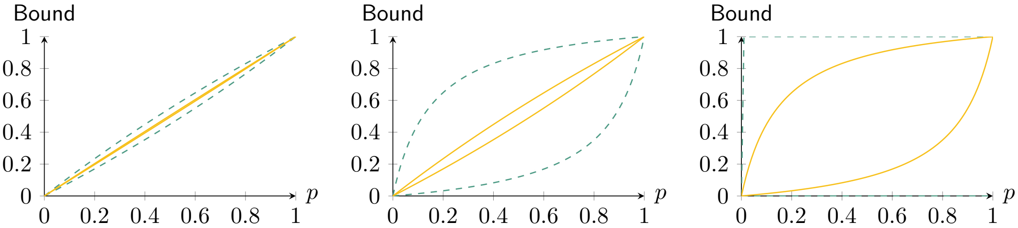
$\downarrow$



$Sal(E)$	Rev	$\phi_1'(Sal(E), Rev)$
high	high	$\varphi_1^* = (\varphi_1 + \varphi_1') \,/\, 2$
high	low	$\varphi_2^* = (\varphi_2 + \varphi_2') \,/\, 2$
low	high	$\varphi_3^* = (\varphi_3 + \varphi_3') \,/\, 2$
low	low	$\varphi_4^* = (\varphi_4 + \varphi_4') \,/\, 2$

Bounding the Change in Query Results

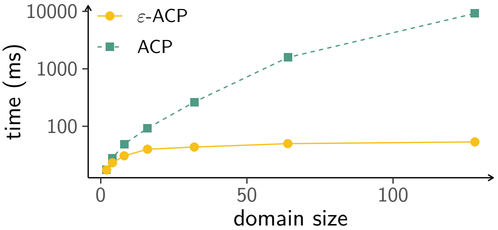
- Bounds for $m = 10$ (left), $m = 100$ (middle), and $m = 1000$ (right) factors
- Dashed line: $\varepsilon = 0.01$, solid line: $\varepsilon = 0.001$
- x-axes depict the original probability p, y-axes reflect the bound on the change in p

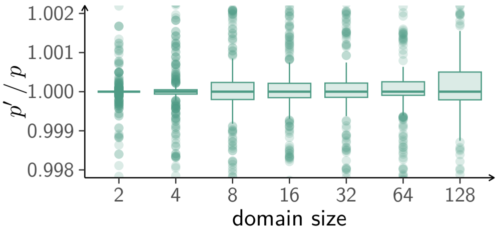


- Bounds apply to arbitrary queries and factor graphs

Experiments

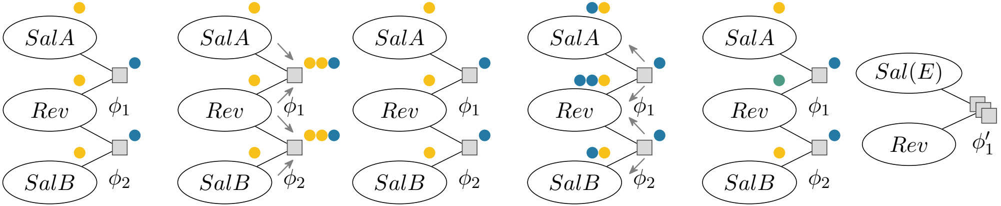
- ε-ACP is a generalisation of ACP that assigns identical colours to groups of ε-equivalent factors (instead of equivalent factors)
- Left: Comparison of run times for lifted inference
- Right: Quotients of query results p' in the modified factor graph and p in the original factor graph





Summary

- Limited practical applicability of ACP solved
- Hyperparameter ε to control the trade-off between exactness and compactness
- Theoretical guarantees for the change in query results



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$Sal(E)$	Rev	$\phi'_1(Sal(E), Rev)$
high	high	$(\varphi_1 + \varphi'_1) / 2$
high	low	$(\varphi_2 + \varphi'_2) / 2$
low	high	$(\varphi_3 + \varphi'_3) / 2$
low	low	$(\varphi_4 + \varphi'_4) / 2$

References

- Babak Ahmadi, Kristian Kersting, Martin Mladenov, and Sriraam Natarajan (2013). »Exploiting Symmetries for Scaling Loopy Belief Propagation and Relational Training«. *Machine Learning* 92, pp. 91–132.
- Kristian Kersting, Babak Ahmadi, and Sriraam Natarajan (2009). »Counting Belief Propagation«. *Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence (UAI-2009)*. AUAI Press, pp. 277–284.
- Malte Luttermann, Tanya Braun, Ralf Möller, and Marcel Gehrke (2024). »Colour Passing Revisited: Lifted Model Construction with Commutative Factors«. *Proceedings of the Thirty-Eighth AAAI Conference on Artificial Intelligence (AAAI-2024)*. AAAI Press, pp. 20500–20507.