On the Detection of Commutative Factors in Factor Graphs: Necessary and Sufficient Conditions

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What is this Work About?

- Probabilistic inference in a Bayesian network or factor graph is intractable in general (Cooper, 1990).
- **Lifted inference** exploits *indistinguishability* of objects for tractable inference (Niepert and Van den Broeck, 2014).
- Constructing a lifted model requires the identification of commutative factors.

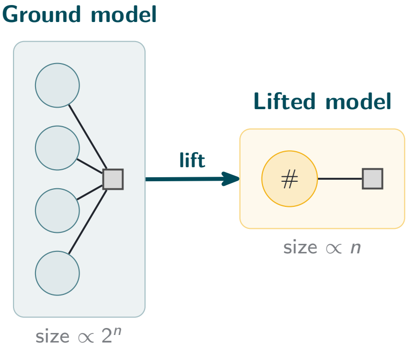
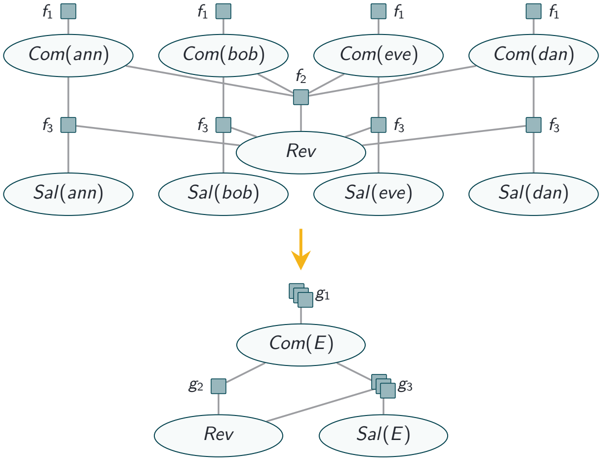


Illustration of Lifted Model Construction

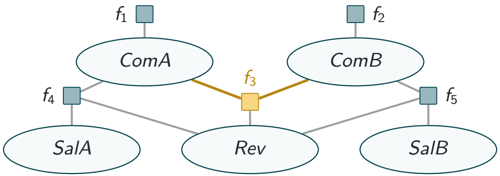
- **Ground model:** random variables per object; repeated structure is not exploited.
- **Lifted model:** groups via *parameterised* random variables and *parametric* factors.
- Size grows with the number of *groups*, not *objects*.



Commutative Factors: Example

- Revenue depends on the number of competent employees, not *which* specific employees are competent.
- Formally, the underlying function is (partially) symmetric:

$$\phi_3(high, low, \cdot) = \phi_3(low, high, \cdot)$$

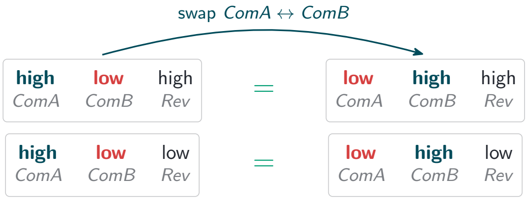


Commutative Factors: Definition and Goal

Commutative Factor

A factor $\phi(R_1, \dots, R_n)$ is **commutative** w.r.t. $\boldsymbol{C} \subseteq \{R_1, \dots, R_n\}$ if its output is invariant under *every* permutation of the arguments in $\boldsymbol{C}$:

$$\phi(r_1, \dots, r_n) = \phi(r_{\pi(1)}, \dots, r_{\pi(n)}) \quad \text{for all } \pi \text{ fixing } R_i \notin \boldsymbol{C}.$$



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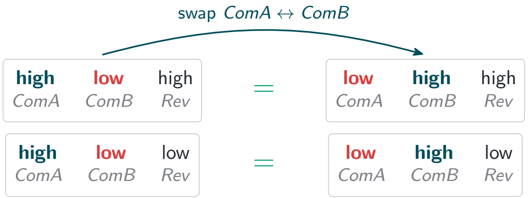

Goal

Find a **maximum sized** commutative subset $\boldsymbol{C}$.

Pruning the Search with Buckets

- Each assignment belongs to a *bucket*.
- Permutations of an assignment fall into the same *bucket*.

ComA	*ComB*	*Rev*	ϕ_3	b
high	*high*	*high*	φ_1	$[3, 0]$
high	*high*	*low*	φ_2	$[2, 1]$
high	*low*	*high*	φ_3	$[2, 1]$
high	*low*	*low*	φ_4	$[1, 2]$
low	*high*	*high*	φ_3	$[2, 1]$
low	*high*	*low*	φ_4	$[1, 2]$
low	*low*	*high*	φ_5	$[1, 2]$
low	*low*	*low*	φ_6	$[0, 3]$

b	$\phi_3^{\succ}(b)$
$[3, 0]$	$\langle \varphi_1 \rangle$
$[2, 1]$	$\langle \varphi_2, \varphi_3, \varphi_3 \rangle$
$[1, 2]$	$\langle \varphi_4, \varphi_4, \varphi_5 \rangle$
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Idea (Luttermann et al., 2024)

Groups of *identical potentials* within a bucket expose candidate arguments.

The Detection of Commutative Factors (DECOR) Algorithm

(Luttermann et al., 2024)

- Idea: Element-wise intersection yields commutative arguments:

$$
\begin{aligned}
&(\{high\}, \{low\}, \{high\}) \\
\cap\ &(\{low\}, \{high\}, \{high\}) \\
=\ &(\emptyset, \emptyset, \{high\})
\end{aligned}
$$

$ComA$	$ComB$	Rev	ϕ_3	b
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$high$	low	low	φ_4	$[1, 2]$
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[1, 2]	$\langle \varphi_4, \varphi_4, \varphi_5 \rangle$
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Flaw in the State of the Art

The element-wise intersection was used as a *sufficient* condition for commutativity, but it is only *necessary*.

Flaw in the State of the Art

$\phi(R_1, R_2, R_3, R_4)$, all ranges $\{high, low\}$:

$\phi(low, low, high, high) = \varphi_1$
$\phi(high, high, low, low) = \varphi_1$
$\phi(\boldsymbol{r}) = \varphi_2 \neq \varphi_1$ for all other $\boldsymbol{r}$

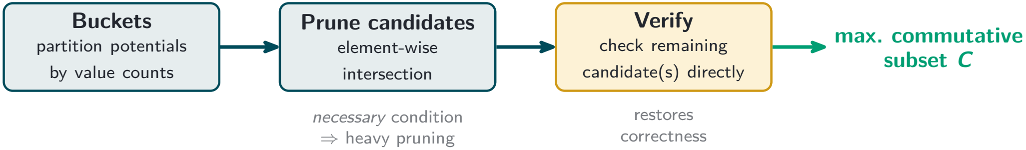
b	$\phi^{\succ}(b)$
$[4, 0]$	$\langle \varphi_2 \rangle$
$[3, 1]$	$\langle \varphi_2, \varphi_2, \varphi_2, \varphi_2 \rangle$
$[2, 2]$	$\langle \varphi_1, \varphi_2, \varphi_2, \varphi_2, \varphi_2, \varphi_1 \rangle$
$[1, 3]$	$\langle \varphi_2, \varphi_2, \varphi_2, \varphi_2 \rangle$
$[0, 4]$	$\langle \varphi_2 \rangle$

Counterexample

The intersection yields $\boldsymbol{C} = \{R_1, R_2, R_3, R_4\}$, yet ϕ is **not** commutative w.r.t. $\boldsymbol{C}$ as, e.g., $\phi(low, high, high, low) = \varphi_2 \neq \varphi_1 = \phi(high, high, low, low)$.

$\Rightarrow$ DECOR might return wrong results.

DECOR+: Prune, Then Verify



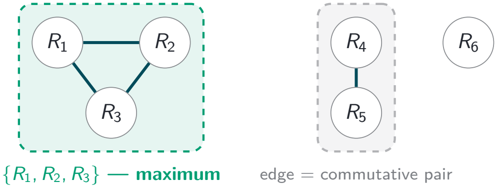
✔ **Correct:** verification directly checks the definition on the few remaining candidates.

✔ **Efficient:** in practice the pruning often leaves *one true* candidate — correctness is essentially for free.

A-DECOR: A Bottom-Up Alternative

Two structural properties of commutativity:

- **Downward closed:** every subset of a commutative set is commutative.
- **Transitive:** C is commutative $\iff$ *every pair* in C is commutative.



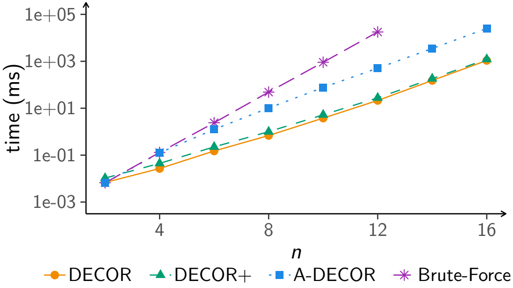
$\Rightarrow$ build commutative sets from *pairs* — the connected components of the *commutativity graph*.

Worst-Case Bound

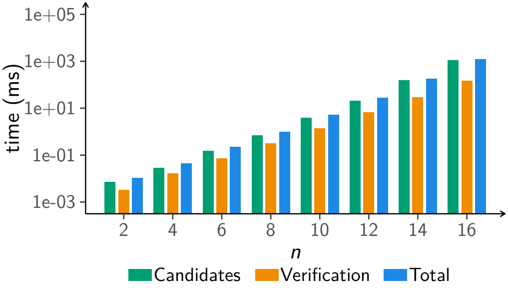
Only $O(n^2)$ commutativity checks — compared to $O(2^n)$ for DECOR+.

But: $\Omega(n^2)$ checks compared to $\Omega(1)$ for DECOR+.

Experiments



Average run time vs. number of arguments n.

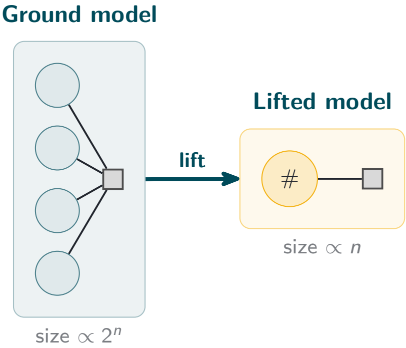


DECOR+: candidates vs. verification.

- DECOR+ **matches the (unsound) DECOR** — correctness comes for free.
- Verification is negligible: pruning almost always leaves a single (true) candidate.

Conclusion

- Identified a flaw in DECOR: a necessary condition was used as a sufficient one.
- DECOR+ to efficiently identify commutative arguments — **correct** and **efficient**.
- A-DECOR: bottom-up, $O(n^2)$ checks — tighter worst-case bound.



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- Cooper, Gregory F. (1990). "The Computational Complexity of Probabilistic Inference using Bayesian Belief Networks". In: *Artificial Intelligence* 42, pp. 393–405.
- Luttermann, Malte, Johann Machemer, and Marcel Gehrke (2024). "Efficient Detection of Commutative Factors in Factor Graphs". In: *Proceedings of the Twelfth International Conference on Probabilistic Graphical Models (PGM-2024)*. PMLR, pp. 38–56.
- Niepert, Mathias and Guy Van den Broeck (2014). "Tractability through Exchangeability: A New Perspective on Efficient Probabilistic Inference". In: *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence (AAAI-2014)*. AAAI Press, pp. 2467–2475.