Towards Learning Differentially Private Probabilistic Relational Models I

AnoMed Seminar

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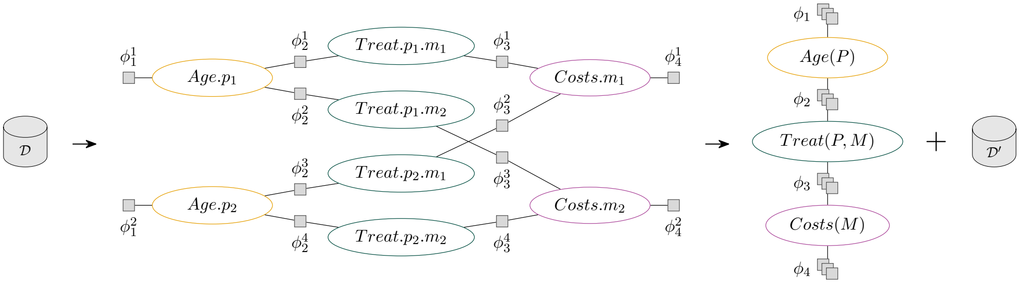
Motivation

- Goal: Learn a differentially private probabilistic relational model (DP PRM)
- Reason over cohorts of patients using a lifted representation
- Sample from the DP PRM to create new publishable datasets

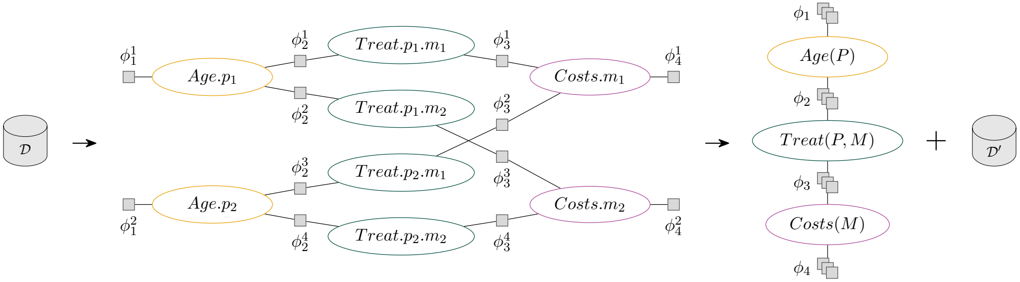


Overview

(Luttermann, Möller, and Hartwig, 2024)

Pipeline to generate synthetic relational data via probabilistic relational models:

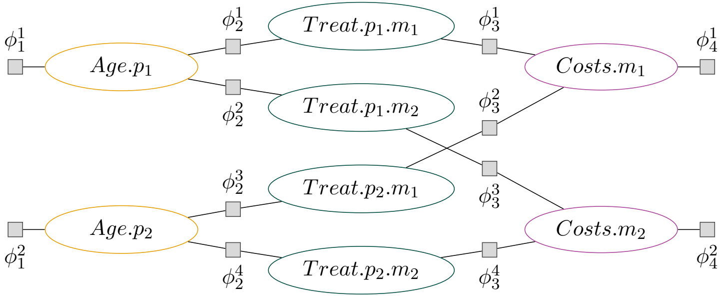
1. Construction of a factor graph
2. Transformation of the factor graph into a PRM (e.g., a parametric factor graph)
3. Sampling the PRM to generate synthetic data samples



Factor Graphs

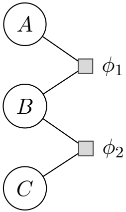
- A factor graph G compactly encodes a full joint probability distribution
- Semantics of G over a set of factors $\boldsymbol{\Phi} = \{\phi_1, \ldots, \phi_m\}$ is given by

$$P_G(\boldsymbol{r}) = \frac{1}{Z} \prod_{j=1}^{m} \phi_j(\boldsymbol{r}_j)$$



Indistinguishability in Factor Graphs

$$P(B) = \sum_{a \in \mathrm{range}(A)} \sum_{c \in \mathrm{range}(C)} P(A = a, B, C = c)$$



A	B	$\phi_1(A,B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

C	B	$\phi_2(C,B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

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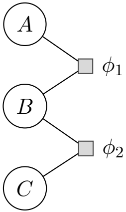

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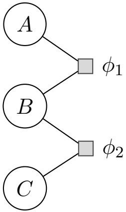
$$
\begin{aligned}
P(B) &= \sum_{a \in \mathrm{range}(A)} \sum_{c \in \mathrm{range}(C)} P(A = a, B, C = c) \\
&= \frac{1}{Z} \cdot \sum_{a \in \mathrm{range}(A)} \sum_{c \in \mathrm{range}(C)} \phi_1(A = a, B) \cdot \phi_2(C = c, B)
\end{aligned}
$$

Indistinguishability in Factor Graphs



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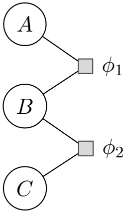
&= \frac{1}{Z} \cdot \sum_{a \in \mathrm{range}(A)} \phi_1(A = a, B) \cdot \sum_{c \in \mathrm{range}(C)} \phi_2(C = c, B)
\end{aligned}
$$

Indistinguishability in Factor Graphs



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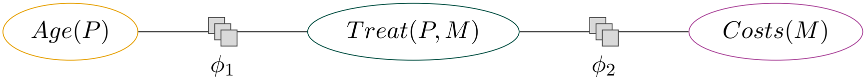
$$
\begin{aligned}
P(B) &= \sum_{a \in \mathrm{range}(A)} \sum_{c \in \mathrm{range}(C)} P(A=a, B, C=c) \\
&= \frac{1}{Z} \cdot \sum_{a \in \mathrm{range}(A)} \sum_{c \in \mathrm{range}(C)} \phi_1(A=a, B) \cdot \phi_2(C=c, B) \\
&= \frac{1}{Z} \cdot \sum_{a \in \mathrm{range}(A)} \phi_1(A=a, B) \cdot \sum_{c \in \mathrm{range}(C)} \phi_2(C=c, B) \\
&= \frac{1}{Z} \cdot \Big(\sum_{a \in \mathrm{range}(A)} \phi_1(A=a, B) \Big)^2 = \frac{1}{Z} \cdot \Big(\sum_{c \in \mathrm{range}(C)} \phi_2(C=c, B) \Big)^2
\end{aligned}
$$

Probabilistic Relational Models: Parametric Factor Graphs

- Compact encoding of a joint probability distribution on a lifted (first-order) level
- Use a representative of indistinguishable objects for computations
- Logical variables to represent groups of random variables
  - $\mathrm{dom}(P) = \{p_1, p_2, \ldots\}$
  - $\mathrm{dom}(M) = \{m_1, m_2, \ldots\}$



Probabilistic Relational Models: Parametric Factor Graphs

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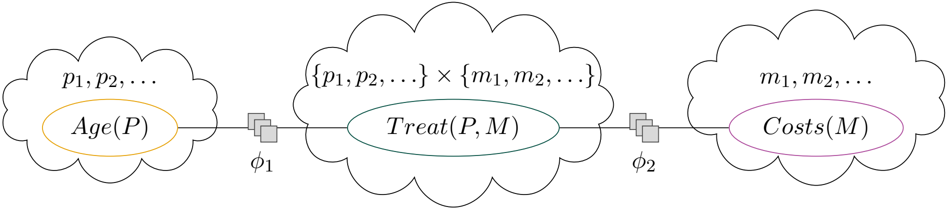
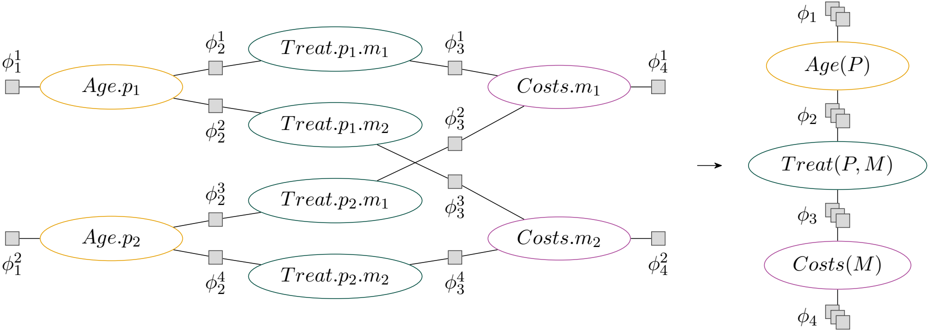


Constructing a Lifted Representation

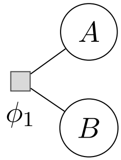
- Idea: Detect symmetric subgraphs by passing colours around
- Assign colours to random variables depending on their ranges and evidence
- Assign colours to factors depending on their potentials



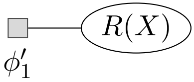
Commutative Factors

- Use histograms to detect commutativity of factors
- Each histogram must be mapped to a unique value
- If a factor is commutative, its neighbours might be grouped

A	B	$\phi_1(A, B)$
true	true	φ_1
true	false	φ_2
false	true	φ_2
false	false	φ_3

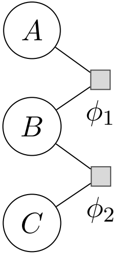


$\#_X[R(X)]$	$\phi'_1(\#_X[R(X)])$
$[2, 0]$	φ_1
$[1, 1]$	φ_2
$[0, 2]$	φ_3



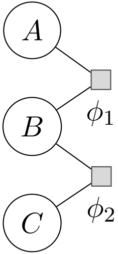
Exchangeable Factors

- Both factor graphs entail equivalent semantics
- Histograms can be used as a filter condition to find permutations



A	B	$\phi_1(A, B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

C	B	$\phi_2(C, B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

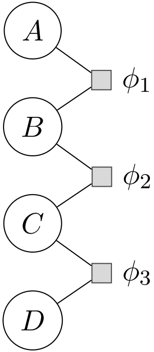


A	B	$\phi_1(A, B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

B	C	$\phi_2(B, C)$
true	true	φ_1
true	false	φ_3
false	true	φ_2
false	false	φ_4

The Advanced Colour Passing Algorithm

(Luttermann, Braun, et al., 2024)



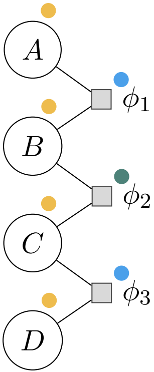
A	B	$\phi_1(A, B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

C	D	$\phi_3(C, D)$
true	true	φ_1
true	false	φ_3
false	true	φ_2
false	false	φ_4

B	C	$\phi_2(B, C)$
true	true	φ_5
true	false	φ_6
false	true	φ_6
false	false	φ_7

The Advanced Colour Passing Algorithm

(Luttermann, Braun, et al., 2024)



A	B	$\phi_1(A, B)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

D	C	$\phi_3(D, C)$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

B	C	$\phi_2(B, C)$
true	true	φ_5
true	false	φ_6
false	true	φ_6
false	false	φ_7

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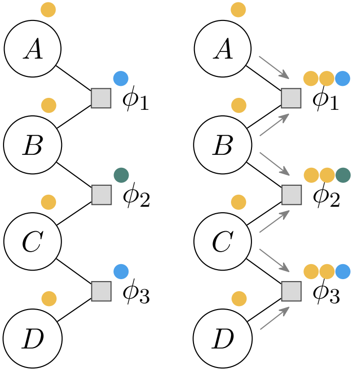

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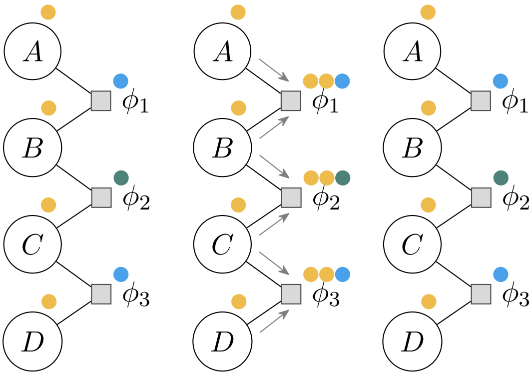

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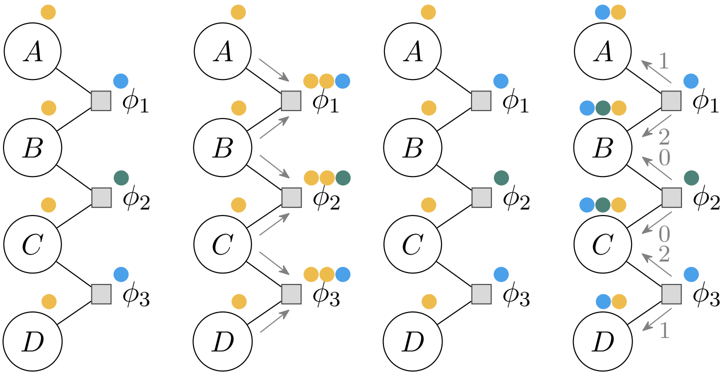

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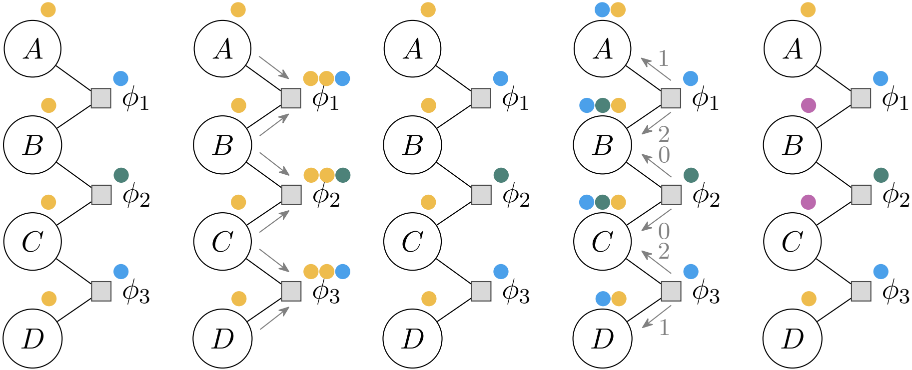

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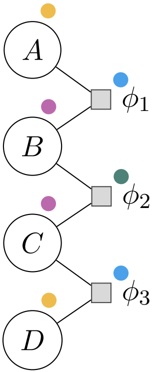
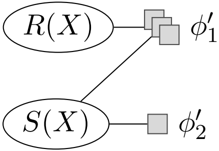




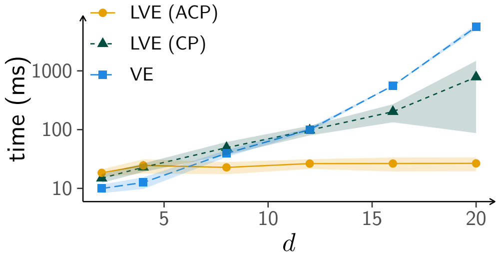
$R(X)$	$S(X)$	$\phi'_1(R(X), S(X))$
true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

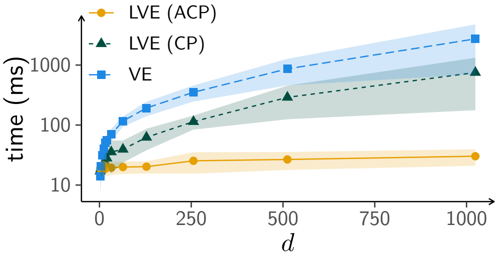
$\#_X[S(X)]$	$\phi'_2(\#_X[S(X)])$
$[2, 0]$	φ_5
$[1, 1]$	φ_6
$[0, 2]$	φ_7

The Advanced Colour Passing Algorithm – Experiments

(Luttermann, Braun, et al., 2024)

- Comparison of run times for lifted inference
- d is the domain size and controls the size of the input factor graph
- Left: Factor graphs with 1 commutative factor (no permuted argument orders)
- Right: Factor graphs where the arguments of 3 percent of the factors are permuted (no commutative factors)

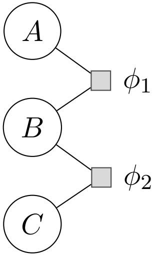




Approximate Lifted Model Construction

(Luttermann, Speller, et al., 2025)

- So far: Strict equality between potentials
  - $\varphi_1 = \varphi'_1$, $\varphi_2 = \varphi'_2$, $\varphi_3 = \varphi'_3$, $\varphi_4 = \varphi'_4$
- Next: Allow for a small deviation between potentials for practical applicability
  - $\varphi_1 \approx \varphi'_1$, $\varphi_2 \approx \varphi'_2$, $\varphi_3 \approx \varphi'_3$, $\varphi_4 \approx \varphi'_4$



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true	true	φ_1
true	false	φ_2
false	true	φ_3
false	false	φ_4

C	B	$\phi_2(C, B)$
true	true	φ'_1
true	false	φ'_2
false	true	φ'_3
false	false	φ'_4

ε-Equivalence

(Luttermann, Speller, et al., 2025)

- Potentials $\varphi_1 \in \mathbb{R}^+$ and $\varphi_2 \in \mathbb{R}^+$ are ε-*equivalent* if

$$\varphi_1 \in [\varphi_2 \cdot (1-\varepsilon), \varphi_2 \cdot (1+\varepsilon)], \text{ and}$$
$$\varphi_2 \in [\varphi_1 \cdot (1-\varepsilon), \varphi_1 \cdot (1+\varepsilon)]$$

- Factors $\phi_1(R_1, \ldots, R_n)$ and $\phi_2(R'_1, \ldots, R'_n)$ are ε-*equivalent* if all potentials in their potential tables are ε-equivalent
- E.g., with $\varepsilon = 0.1$, $\phi_1(A, B)$ and $\phi_2(C, B)$ are ε-equivalent:

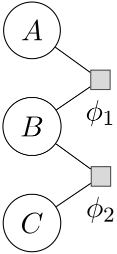
A	B	$\phi_1(A, B)$
true	true	0.8
true	false	0.3
false	true	0.5
false	false	0.2

C	B	$\phi_2(C, B)$
true	true	0.84
true	false	0.31
false	true	0.51
false	false	0.22

Grouping ε-Equivalent Factors

(Luttermann, Speller, et al., 2025)

- Potential tables must be identical to exploit exponentiation
- How to choose φ_i^*?



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true	true	φ'_1
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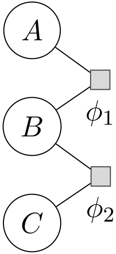

A	B	$\phi_1(A, B)$
true	true	φ_1^*
true	false	φ_2^*
false	true	φ_3^*
false	false	φ_4^*

C	B	$\phi_2(C, B)$
true	true	φ_1^*
true	false	φ_2^*
false	true	φ_3^*
false	false	φ_4^*

Grouping ε-Equivalent Factors

(Luttermann, Speller, et al., 2025)

For a group of pairwise ε-equivalent factors $\boldsymbol{G} = \{\phi_1, \ldots, \phi_k\}$, we want to set

$$\phi_1 = \phi^*, \ldots, \phi_k = \phi^*$$

such that

$$\phi^* = \arg\min_{\phi_j} \sum_{\phi_i \in \boldsymbol{G}} Err(\phi_i, \phi_j),$$

where $Err(\phi_i, \phi_j)$ is the sum of squared deviations between ϕ_i's and ϕ_j's potentials:

$$Err(\phi_i, \phi_j) = \sum_{r_1, \ldots, r_n} \Big(\phi_i(r_1, \ldots, r_n) - \phi_j(r_1, \ldots, r_n)\Big)^2$$

Grouping ε-Equivalent Factors

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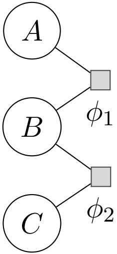
Thus, ϕ^* is chosen as the arithmetic mean of the potentials in $\boldsymbol{G}$:

$$\phi^*(\boldsymbol{r}) = \frac{1}{k} \sum_{i=1}^{k} \phi_i(\boldsymbol{r})$$

Grouping ε-Equivalent Factors

(Luttermann, Speller, et al., 2025)

- Choose φ_i^* as the row-wise arithmetic mean of the potentials



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false	true	φ_3
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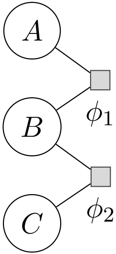
C	B	$\phi_2(C,B)$
true	true	φ_1'
true	false	φ_2'
false	true	φ_3'
false	false	φ_4'



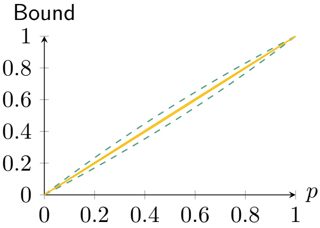
A	B	$\phi_1(A,B)$
true	true	$(\varphi_1 + \varphi_1') \ / \ 2$
true	false	$(\varphi_2 + \varphi_2') \ / \ 2$
false	true	$(\varphi_3 + \varphi_3') \ / \ 2$
false	false	$(\varphi_4 + \varphi_4') \ / \ 2$

C	B	$\phi_2(C,B)$
true	true	$(\varphi_1 + \varphi_1') \ / \ 2$
true	false	$(\varphi_2 + \varphi_2') \ / \ 2$
false	true	$(\varphi_3 + \varphi_3') \ / \ 2$
false	false	$(\varphi_4 + \varphi_4') \ / \ 2$

Bounding the Change in Query Results

(Luttermann, Speller, et al., 2025)

- Bounds for $m = 10$ (left), $m = 100$ (middle), and $m = 1000$ (right) factors
- Dashed line: $\varepsilon = 0.01$, solid line: $\varepsilon = 0.001$
- x-axes depict the original probability p, y-axes reflect the bound on the change in p

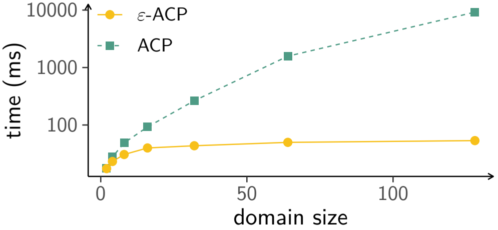


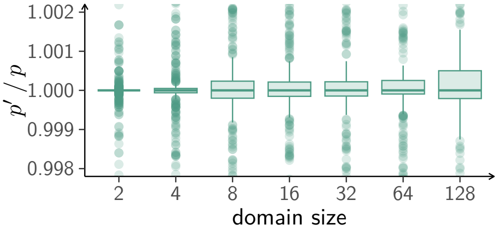
- Bounds apply to arbitrary queries and factor graphs

The ε-Advanced Colour Passing Algorithm – Experiments

(Luttermann, Speller, et al., 2025)

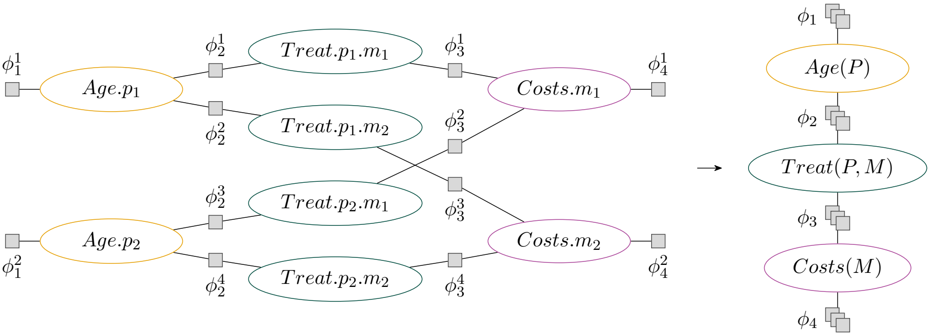
- ε-ACP is a generalisation of the Advanced Colour Passing Algorithm that assigns identical colours to groups of ε-equivalent factors (instead of equivalent factors)
- Left: Comparison of run times for lifted inference
- Right: Quotients of query results p' in the modified factor graph and p in the original factor graph





Conclusion

- First steps towards learning differentially private probabilistic relational models
- Problem of learning a (non-DP) PRM from a propositional factor graph solved



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