UNIVERSITÄT ZU LÜBECK

Practical Algorithms for Orientations of Partially Directed Graphical Models

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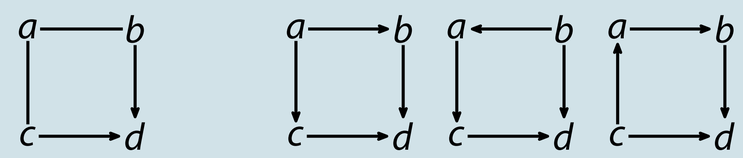
1. Problem Setting

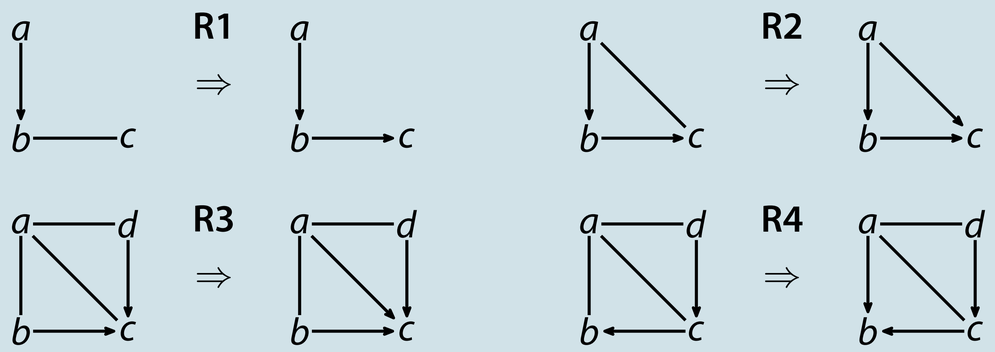
Input: A partially directed acyclic graph (PDAG) G.

Output: Any consistent DAG extension of G or $\perp$ if no such extension exists.

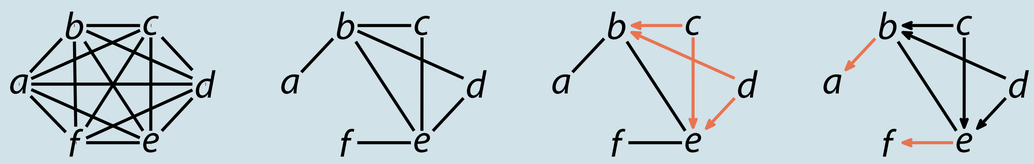


PDAG | Markov Equivalence Class

Application: Maximal orientation (applying the four Meek rules exhaustively to a PDAG) by (i) extending the PDAG into a DAG and (ii) computing the maximal orientation efficiently utilizing the topological ordering of the DAG.



Maximal orientations are an important building block in causal discovery, e. g., in the final step of the PC algorithm.



Complete graph | Skeleton | v-structures | Meek rules

2. Previous Work

Algorithm	Complexity	Advantage
Dor and Tarsi (1992) (DT)	$O(n^4)$	Less practical overhead
Wienöbst et al. (2021) (WBL)	$O(n^3)$	Less repeated calculations

Our goal: Combine the advantages of both algorithms to obtain an efficient algorithm for practical applications.

3. The Dor-Tarsi Algorithm

Prerequisite: Concept of a potential-sink, that is, a vertex v

- without outgoing edges and
- all vertices x with $v - x$ being adjacent to all other neighbors of v.

Dor-Tarsi Algorithm to extend a PDAG G:

1. $D \leftarrow$ copy of G
2. Repeat n times:
   (i) $v \leftarrow$ any potential-sink in G; if no potential-sink exists return $\perp$
   (ii) Remove v and its incident edges from G
   (iii) Direct all undirected edges incident to v towards v in D
3. Return D

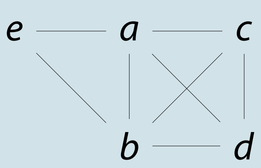
4. Two New Simple Algorithms for Extendability

Building on the Dor-Tarsi (DT) algorithm with simple modifications:

(i) DT with heuristic (DTH): Iterate over vertices in order of increasing degree when searching for a potential-sink

(ii) Heuristic with additional caching and improved time complexity expected in $O(n^3)$ (DTIC)

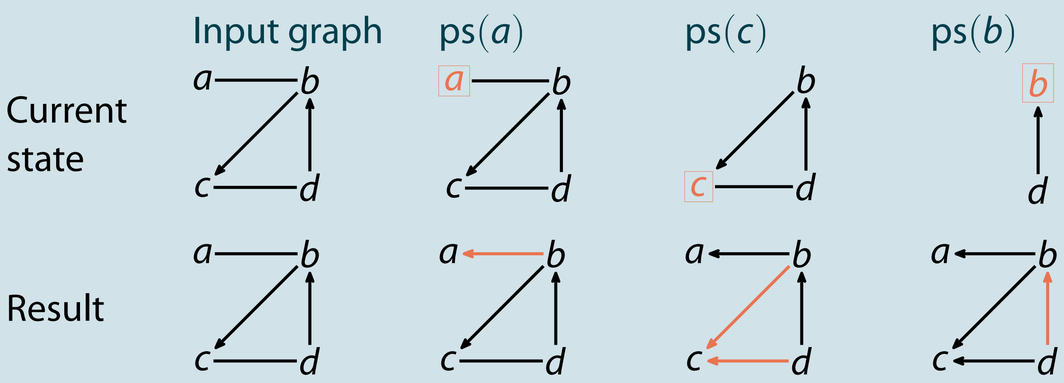
PDAG G | #Adj. tests for the first potential-sink (ps)



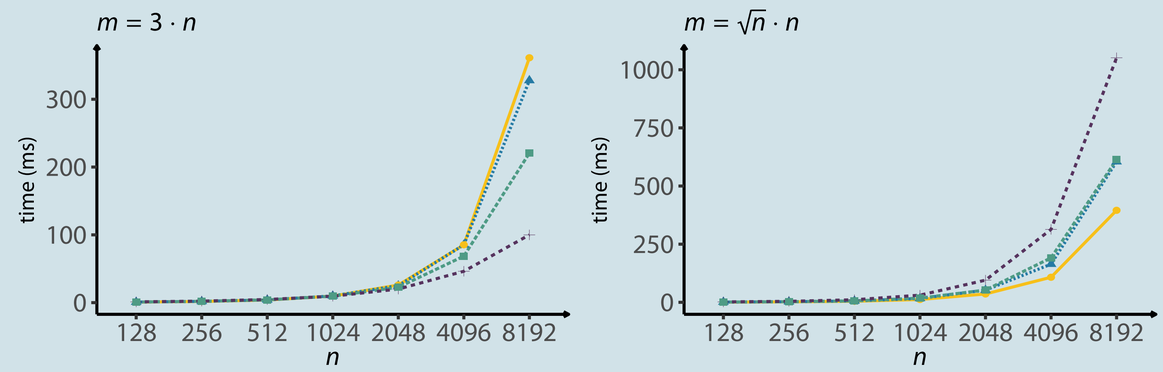
DT: $b \overset{?}{\sim} c, b \overset{?}{\sim} d, b \overset{?}{\sim} e, c \overset{?}{\sim} d, c \overset{?}{\sim} e$ ps(a) ✗
$a \overset{?}{\sim} c, a \overset{?}{\sim} d, a \overset{?}{\sim} e, c \overset{?}{\sim} d, c \overset{?}{\sim} e$ ps(b) ✗
$a \overset{?}{\sim} b, a \overset{?}{\sim} d, b \overset{?}{\sim} d$ ps(c) ✓

DTH: $a \overset{?}{\sim} b$ ps(e) ✓

5. Example Run of the New Algorithm



6. Evaluation of Extension Algorithms



7. Application to Maximal Orientations

- DIRECT-MEEK: Apply Meek's rules directly in a while-loop.
- CE-MEEK: Compute a consistent extension, find the corresponding CPDAG, and then apply Meek's rules in a single iteration.